ON THE DETERMINATION OF THE NONAXISYMMETRIC TEMPERATURE FIELD OF AN ORTHOTROPIC HOLLOW CYLINDER AND SPHERE

(K OPREDELENIIU NEOSESIMMETRICHNOGO TEMPERATURNOGO POLIA ORTOTROPNOGO POLOGO TSILINDRA I SHARA)

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Problems on the determination of symmetric temperature fields for both a solid and hollow isotropic cylinder and sphere under different initial and boundary conditions have been examined in a number of works ([1 and 2], etc.).

Of considerable interest is the determination of nonaxisymmetric nonstationary temperature fields for an isotropic and orthotropic hollow cylinder and sphere under various boundary conditions corresponding to the heat exchange on the outer and inner surfaces.

In solving these problems, the desired temperature functions may be represented as series (trigonometric series along the generator and directrix in the cylinder case; trigonometric series along the parallels and Legendre polynomials along the meridian in the sphere case).

Then, if the Laplace transform is used, finding the transforms of the expansion coefficients may be reduced to integration of the auxiliary Bessel equation. After having integrated this auxiliary equation by application of inversion theorems for the Laplace transform (taking into account the transformed boundary conditions), the expansion coefficients can be determined, and then the desired temperature function as well.

When the boundary conditions on the surfaces of an orthotropic hollow cylinder or sphere are given in the form of linear combinations of the temperature function and its first derivative with respect to the normal to the surface, it is necessary to determine and investigate the roots of the transcendental equation

$$[a_1 J_\nu(x) - b_1 x J_{\nu+1}(x)]\,[a_2 N_\nu(\mu x) - b_2 \mu x N_{\nu+1}(\mu x)] -$$
$$- [a_2 J_\nu(\mu x) - b_2 \mu x J_{\nu+1}(\mu x)]\,[a_1 N_\nu(x) - b_1 x N_{\nu+1}(x)] = 0 \qquad (\infty > \mu > 1,\ \nu \geqslant 0) \quad (0.1)$$

in order to utilize theorems to evaluate the contour integrals and to prove the final results.

McMahon [3], Sasaki [4], Carslaw [5], Lipow and Zwick [6], etc. considered particular cases of this equation (for $\nu = 0$ and $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$, for $\nu = 0$, $a_2 = 0$, etc.). The results of [3] are used below to evaluate and investigate the roots of Equation (0.1).

1. Let us examine (0.1) in the case when $b_1 b_2 \neq 0$. Following McMahon, let us utilize asymptotic expansions of the Bessel functions [7]

$$a_1 J_\nu(x) - b_1 x J_{\nu+1}(x) = -R_1 \sqrt{2/\pi x}\sin(x - \pi\nu/2 - \pi/4 - \theta_1)$$
$$a_1 N_\nu(x) - b_1 x N_{\nu+1}(x) = R_1 \sqrt{2/\pi x}\cos(x - \pi\nu/2 - \pi/4 - \theta_1) \quad (1.1)$$

$$a_2 J_\nu(\mu x) - b_2 \mu x J_{\nu+1}(\mu x) = -R_2 \sqrt{2/\pi\mu x}\sin(\mu x - \pi\nu/2 - \pi/4 - \theta_2)$$
$$a_2 N_\nu(\mu x) - b_2 \mu x N_{\nu+1}(\mu x) = R_2 \sqrt{2/\pi\mu x}\cos(\mu x - \pi\nu/2 - \pi/4 - \theta_2) \quad (1.2)$$

to determine the roots of the transcendental equation (0.1).

Here

$$R_1\cos\theta_1 = b_1 x + \sum_{k=0}^{n} \frac{(-1)^k}{(2x)^{2k+1}} \frac{1}{(2k+1)!} \frac{\Gamma(\nu+2k+3/2)}{\Gamma(\nu-2k-1/2)}\left[a_1 - \frac{b_1}{4(k+1)} \times \times \left(\nu+2k+\frac{5}{2}\right)\left(\nu+2k+\frac{3}{2}\right)\right] \quad (1.3)$$

$$R_1\sin\theta_1 = \sum_{k=0}^{n} \frac{(-1)^k}{(2x)^{2k}} \frac{1}{(2k)!} \frac{\Gamma(\nu+2k+1/2)}{\Gamma(\nu-2k+1/2)}\left[a_1 - \frac{b_1}{2(2k+1)} \times \times \left(\nu+2k+\frac{3}{2}\right)\left(\nu+2k+\frac{1}{2}\right)\right]$$

and we obtain the values of $R_2\cos\theta_2$ and $R_2\sin\theta_2$ from (1.3) by replacing a_1 by a_2, b_1 by b_2, x by μx therein.

Inserting (1.1) and (1.2) into the original equation (0.1), we obtain [3]

$$\sin[x(\mu-1) - \theta_2 - \theta_1] = 0, \quad \text{for} \quad x(\mu-1) - \theta_2 + \theta_1 = s\pi \quad (1.4)$$

From (1.3) the values of $\tan\theta_1$ and $\tan\theta_2$ may be represented as

$$\tan\theta_1 = \sum_{k=0}^{\infty} A_{2k+1}\frac{1}{(8x)^{2k+1}}, \qquad \tan\theta_2 = \sum_{k=0}^{\infty} B_{2k+1}\frac{1}{(8\mu x)^{2k+1}} \quad (1.5)$$

Here (*)

$$A_1 = 8\frac{a_1}{b_1} - 4\left(\nu+\frac{1}{2}\right)\left(\nu+\frac{3}{2}\right), \qquad A_3 = -64\frac{\Gamma(\nu+5/2)}{\Gamma(\nu-3/2)}\left[\frac{a_1}{b_1} - \frac{1}{6}\left(\nu+\frac{5}{2}\right) \times \times \left(\nu+\frac{7}{2}\right)\right] - 256\frac{\Gamma(\nu+3/2)}{\Gamma(\nu-1/2)}\left[\frac{a_1}{b_1} - \frac{1}{2}\left(\nu+\frac{1}{2}\right)\left(\nu+\frac{3}{2}\right)\right] \times \times \left[\frac{a_1}{b_1} - \frac{1}{4}\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{5}{2}\right)\right]$$

$$A_5 = \frac{256}{3}\frac{\Gamma(\nu+9/2)}{\Gamma(\nu-7/2)}\left(\frac{a_1}{b_1} - \frac{(\nu+9/2)(\nu+11/2)}{10}\right) + + 2048\frac{\Gamma(\nu+3/2)\Gamma(\nu+5/2)}{\Gamma(\nu-1/2)\Gamma(\nu-3/2)}\left(\frac{a_1}{b_1} - \frac{(\nu+3/2)(\nu+5/2)}{4}\right)\left(\frac{a_1}{b_1} - \frac{(\nu+5/2)(\nu+7/2)}{6}\right) + + \frac{2048}{3}\frac{\Gamma(\nu+7/2)}{\Gamma(\nu-5/2)}\left(\frac{a_1}{b_1} - \frac{(\nu+1/2)(\nu+3/2)}{2}\right)\left(\frac{a_1}{b_1} - \frac{(\nu+7/2)(\nu+9/2)}{8}\right) + + 8192\left[\frac{\Gamma(\nu+3/2)}{\Gamma(\nu-1/2)}\right]^2\left(\frac{a_1}{b_1} - \frac{(\nu+1/2)(\nu+3/2)}{2}\right)\left(\frac{a_1}{b_1} - \frac{(\nu+3/2)(\nu+5/2)}{4}\right)^2$$

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*) It is not possible to obtain a general expression for A_{2k+1}. Values are presented only for the first three coefficients A_1, A_3, A_5. The determination of the value of each succeeding coefficient from (1.3) is not difficult in principle, however, there are extremely tedious calculations associated with it.

We obtain the values of B_j by replacing a_1/b_1 by a_2/b_2 in the expressions for the A_j .

Having the values of $\tan\theta_1$ and $\tan\theta_2$, we determine the angles θ_1 and θ_2 by means of Formulas

$$\theta_j = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} (\tan\theta_j)^{2k+1}$$

or

$$\theta_1 = A_1 \frac{1}{8x} + \left(A_3 - \frac{A_1^3}{3}\right)\frac{1}{(8x)^3} + \left(A_5 - A_1^2 A_3 + \frac{A_1^5}{5}\right)\frac{1}{(8x)^5} + \cdots$$
$$\theta_2 = B_1 \frac{1}{8\mu x} + \left(B_3 - \frac{B_1^3}{3}\right)\frac{1}{(8\mu x)^3} + \left(B_5 - B_1^2 B_3 + \frac{B_1^5}{5}\right)\frac{1}{(8\mu x)^5} + \cdots \quad (1.6)$$

Inserting (1.6) into (1.4) we obtain

$$x = \frac{s\pi}{\mu - 1} + \frac{B_1 - \mu A_1}{8\mu(\mu-1)}\frac{1}{x} + \frac{3B_3 - B_1^3 - \mu^3(3A_3 - A_1^3)}{3(8\mu)^3(\mu-1)}\frac{1}{x^3} + $$
$$+ \frac{5B_5 - 5B_1^2 B_3 + B_1^5 - \mu^5(5A_5 - 5A_1^2 A_3 + A_1^5)}{5(8\mu)^5(\mu-1)}\frac{1}{x^5} + \cdots$$

whose inversion by utilizing a Lagrange series [8] will yield the final value of the root x_s of the original equation (0.1)

$$x_s = \frac{s\pi}{\mu-1} + \frac{B_1 - \mu A_1}{8\mu\pi s} + \frac{\mu - 1}{3(8\mu\pi s)^3}[(\mu-1)(3B_3 - B_1^3 - 3\mu^3 A_3 + A_1^3\mu^3) - \quad (1.7)$$
$$- 24\mu(B_1 - \mu A_1)^3] + \frac{(\mu-1)^2}{15(8\mu\pi s)^5}[3(\mu-1)^3(5B_5 - 5B_1^2 B_3 + B_1^5 - 5\mu^5 A_5 + 5\mu^5 A_1^2 A_3 -$$
$$- \mu^5 A_1^5) - 160\mu(\mu-1)(B_1 - \mu A_1)(3B_3 - B_1^3 - 3\mu^3 A_3 + A_1^3\mu^3) + 1920\mu^2(B_1 - \mu A_1)^3] + \cdots$$

Omitting analogous computations, let us note that for $b_1 = b_2 = 0$ we should take $A_{2k+1} = B_{2k+1} = C_{2k+1}$, in (1.7), where

$$C_1 = -4(\nu - 1/2)(\nu + 1/2), \qquad C_3 = \frac{32}{3}\frac{\Gamma(\nu + 7/2)}{\Gamma(\nu - 5/2)} - 32(\nu - 1/2)(\nu + 1/2)\frac{\Gamma(\nu + 5/2)}{\Gamma(\nu - 3/2)}$$

$$C_5 = -\frac{128}{15}\frac{\Gamma(\nu + 11/2)}{\Gamma(\nu - 9/2)} + \frac{128}{3}(\nu - 1/2)(\nu + 1/2)\frac{\Gamma(\nu + 9/2)}{\Gamma(\nu - 7/2)} +$$
$$+ 256\frac{\Gamma(\nu + 5/2)}{\Gamma(\nu - 3/2)}\left[\frac{1}{3}\frac{\Gamma(\nu + 7/2)}{\Gamma(\nu - 5/2)} - (\nu - 1/2)(\nu + 1/2)\frac{\Gamma(\nu + 5/2)}{\Gamma(\nu - 3/2)}\right]$$

. . . .

Replacing s in (1.7) by $s + \frac{1}{2}$ and the coefficients A_{2k+1} or B_{2k+1} by C_{2k+1} , we obtain formulas to evaluate the roots x_s of the original equation (0.1) for the $b_1 = 0$, $b_2 \neq 0$ or $b_1 \neq 0$, $b_2 = 0$ cases, respectively.

Let us note that for certain values of the parameters Formula (1.7) may turn out to be unsuitable for the evaluation of the first few roots. In these cases the first roots should be determined either from prepared tables or by the numerical solution of the specific equation. This circumstance is without value in the investigation of the asymptotic behavior of the roots of the original equation (0.1).

2. In the case of a finite value of ν it follows from (1.7) that the roots x_s of the original equation (0.1).tend to infinity, as s tends to infinity, at least as rapidly as ms , where $m \neq 0$.

It also follows from (1.7) that the original equation (0.1) has a nondenumerable set of real roots for real values of the parameters ν, a_1 , a_2, b_1 , b_2, μ .

Let us investigate their behavior as the index ν increases without limit.

Taking into account that the function on the left-hand side of (0.1) is even, we can, without limiting the generality of the results, consider only positive roots ($x_s > 0$).

It is known [7]that in the asymptotic expansion of Bessel functions with large indices it is necessary to examine separately the case when the argument is less than the index ($\mu x/\nu = q$, where $0 \leq q < 1$) , greater than the index, and equals the index.

We represent the Bessel functions for large values of the index (when the argument is less than the index but greater than zero) by utilizing the asymptotic formulas [7]

$$J_\nu(x) = \frac{e^{-\nu(\alpha - \tanh\alpha)}}{\sqrt{2\pi\nu\tanh\alpha}}\left[1 + \frac{1}{\nu}\left(\frac{1}{8}\coth\alpha - \frac{5}{24}\coth^3\alpha\right) + \cdots\right]$$

$$N_\nu(x) = \frac{\sqrt{2}e^{\nu(\alpha - \tanh\alpha)}}{\sqrt{\pi\nu\tanh\alpha}}\left[1 - \frac{1}{\nu}\left(\frac{1}{8}\coth\alpha - \frac{5}{24}\coth^3\alpha\right) + \cdots\right]$$

$$J_\nu(\mu x) = \frac{e^{-\nu(\beta - \tanh\beta)}}{\sqrt{2\pi\nu\tanh\beta}}\left[1 + \frac{1}{\nu}\left(\frac{1}{8}\coth\beta - \frac{5}{24}\coth^3\beta\right) + \cdots\right]$$

$$N_\nu(\mu x) = \frac{\sqrt{2}e^{\nu(\beta - \tanh\beta)}}{\sqrt{\pi\nu\tanh\beta}}\left[1 - \frac{1}{\nu}\left(\frac{1}{8}\coth\beta - \frac{5}{24}\coth^3\beta\right) + \cdots\right]$$

$$\cosh\alpha = \frac{\nu}{x}, \qquad \cosh\beta = \frac{\nu}{\mu x}$$

For Bessel functions with index $\nu + 1$ we shall use the notation

$$\cosh\alpha' = (\nu + 1)\,/\,x, \; \cosh\beta' = (\nu + 1)\,/\,\mu x.$$

Let us note that only positive values of α, β, α', β' are subject to examination; hence $\alpha - \beta > 0,\ \alpha' - \alpha > 0,\ \beta' - \beta > 0.$

Let us show that (0.1) has no roots $x_s > 0$ for which q tends to zero as the index ν grows without limit.

For definiteness, let us assume that none of the coefficients of Equation (0.1) is zero ($a_1 \neq 0,\ b_1 \neq 0,\ a_2 \neq 0,\ b_2 \neq 0$). Cases when one of these coefficients is zero do not differ, in principle, from the case under consideration. Cases when two coefficients are simultaneously zero are either meaningless (the cases $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$, say) or reduce the equation under consideration to equations investigated in [3 to 6].

Utilizing the asymptotic formulas presented above for the Bessel functions, we represent the fundamental equation (0.1) as

$$\frac{a_1 a_2}{\sqrt{\tanh\alpha\tanh\beta}}\{\exp[-\nu(\alpha - \beta + \alpha' - \beta - \tanh\alpha + \tanh\beta - \tanh\alpha' + \tanh\beta) - (\alpha' - \tanh\alpha')] -$$
$$- \exp[-\nu(\alpha' - \alpha - \tanh\alpha' + \tanh\alpha) - (\alpha' - \tanh\alpha')]\} -$$
$$- \frac{a_2 b_1 x\sqrt{\nu}}{\sqrt{(1+\nu)\tanh\alpha'\tanh\beta}}\{\exp[-2\nu(\alpha' - \beta - \tanh\alpha' + \tanh\beta) - 2(\alpha' - \tanh\alpha')] - 1\} -$$
$$- \frac{a_1 b_2 \mu x\sqrt{\nu}}{\sqrt{(1+\nu)\tanh\alpha\tanh\beta'}}\{\exp[-\nu(\alpha - \beta' + \alpha' - \beta - \tanh\alpha + \tanh\beta' - \tanh\alpha' + \tanh\beta) -$$
$$- (\alpha' - \beta' - \tanh\alpha' + \tanh\beta')] - \exp[-\nu(\alpha' - \alpha + \beta' - \beta - \tanh\alpha' + \tanh\alpha -$$
$$- \tanh\beta' + \tanh\beta) - (\alpha' + \beta' - \tanh\alpha' - \tanh\beta')]\} + \frac{b_1 b_2 \mu x^2 \nu}{(1+\nu)\sqrt{\tanh\alpha'\tanh\beta'}}\{\exp[-\nu(3\alpha' - \beta' -$$
$$- 2\beta - \tanh\alpha' + \tanh\beta' - 2\tanh\alpha' + 2\tanh\beta) - (3\alpha' - \beta' - 3\tanh\alpha' + \tanh\beta')] -$$
$$- \exp[-\nu(\beta' - 2\beta + \alpha' - \tanh\beta' - \tanh\alpha' + 2\tanh\beta) - 2(\alpha' - \tanh\alpha')]\} = 0$$

Hence, taking account of the limit relationships for $\nu \to \infty$ and $q \to 0$

$$\lim \alpha' = \lim \beta' = \infty, \qquad \lim (\alpha' - \beta') = \lim (\alpha' - \beta) = \ln \mu$$

$$\lim [\nu (\alpha' - \alpha)] = \lim [\nu (\beta' - \beta)] = 1,$$

$$\lim \tanh \alpha = \lim \tanh \alpha' = \lim \tanh \beta = \lim \tanh \beta' = 1$$

$$\lim [\nu (\alpha - \beta)] = \lim [\nu (\alpha - \beta')] = \infty$$

we obtain $a_2 b_1 x_s = 0$, which contradicts the original assumptions $a_2 \neq 0$, $b_1 \neq 0$, $x_s > 0$. The obtained contradiction permits the assertion that there are no roots among those considered for which q will tend to zero as ν grows without limit.

Thus, even without examining the remaining two cases (when the argument of the Bessel functions is greater than or equal to the index), it can be considered proven that as ν grows without limit the nonzero roots x_s of the original equation (0.1) grow at least as rapidly as $m\nu$, where $m \neq 0$.

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